Chapter 4

Probability and Counting Rules

Chapter 4 Overview

Introduction

- 4-1 Probability of Events
- 4-2 Conditional Probability of Events
- 4-3 Rules of Probability
- 4-4 Mutually Exclusive Events
- 4-5 Independent Events
- 4-6 Are Mutual Exclusiveness and Independence Related?

Probability

- **Probability** can be defined as the chance of an event occurring. It can be used to quantify what the “odds” are that a specific event will occur. Some examples of how probability is used everyday would be weather forecasting, “75% chance of snow” or for setting insurance rates.

4-1 to 4-6 Probability

- A **probability experiment** is a chance process that leads to well-defined results called outcomes.
- An **outcome** is the result of a single trial of a probability experiment.
- A **sample space** is the set of all possible outcomes of a probability experiment.
- An **event** consists of outcomes.

Sample Spaces

Experiment	Sample Space
Toss a coin	Head, Tail
Roll a die	1, 2, 3, 4, 5, 6
Answer a true/false question	True, False
Toss two coins	HH, HT, TH, TT

Ex.) 1 – Sample Space

Find the sample space for spinning a game spinner twice (the spinner is labeled 1-10).

Spin 1	Spin 2									
	1	2	3	4	5	6	7	8	9	10
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(2,10)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)	(3,7)	(3,8)	(3,9)	(3,10)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)	(4,7)	(4,8)	(4,9)	(4,10)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)	(5,7)	(5,8)	(5,9)	(5,10)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)	(6,7)	(6,8)	(6,9)	(6,10)
7	(7,1)	(7,2)	(7,3)	(7,4)	(7,5)	(7,6)	(7,7)	(7,8)	(7,9)	(7,10)
8	(8,1)	(8,2)	(8,3)	(8,4)	(8,5)	(8,6)	(8,7)	(8,8)	(8,9)	(8,10)
9	(9,1)	(9,2)	(9,3)	(9,4)	(9,5)	(9,6)	(9,7)	(9,8)	(9,9)	(9,10)
10	(10,1)	(10,2)	(10,3)	(10,4)	(10,5)	(10,6)	(10,7)	(10,8)	(10,9)	(10,10)

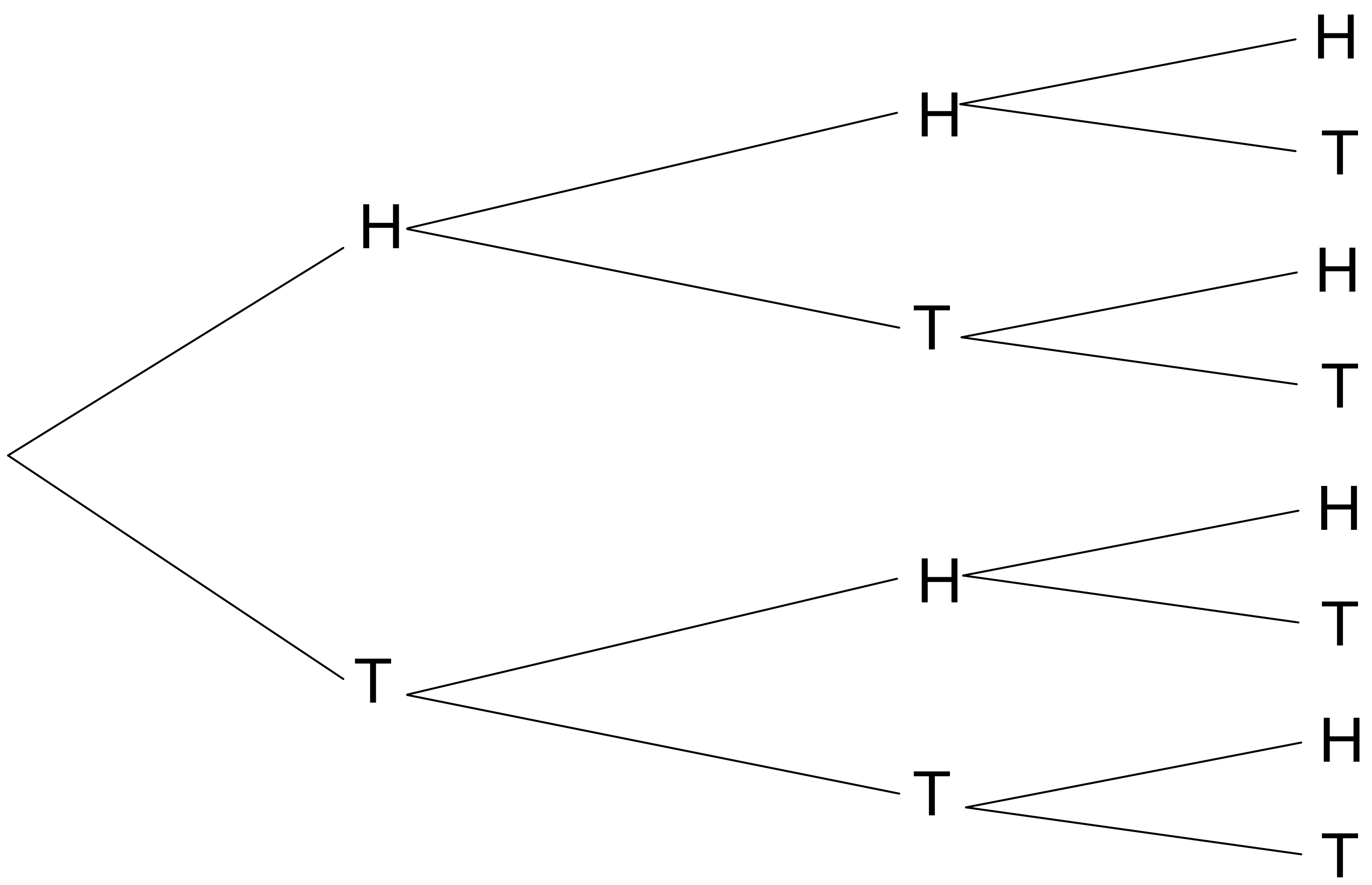
Ex.) 2 – Sample Space

Find the sample space for tossing 3 coins.

HHH HHT HTH HTT THH THT TTH TTT

Ex.) 3 – Sample Space

Use a tree diagram to find the sample space for flipping a coin 3 times.



HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

Sample Space

- What is the sample space for choosing a letter from the word probability?

Class Activity

- Using a tree diagram, write the sample space for selecting 4 colors out of the choices red, blue, and yellow.

Sample Spaces and Probability

There are three basic interpretations of probability:

- **Classical probability**
- **Empirical probability**
- **Subjective probability**

Sample Spaces and Probability

Classical probability uses sample spaces to determine the numerical probability that an event will happen and assumes that all outcomes in the sample space are equally likely to occur.

$$\boxed{P(E) = \frac{n(E)}{n(S)} = \frac{\text{\# of desired outcomes}}{\text{Total \# of possible outcomes}}}$$

Sample Spaces and Probability

Rounding Rule for Probabilities

Probabilities should be expressed as reduced fractions or rounded to two or three decimal places. When the probability of an event is an extremely small decimal, it is permissible to round the decimal to the first nonzero digit after the decimal point.

Ex.) 4 – Classical Probability

If a family has three children, find the probability that one of the three children are girls.

Sample Space:

BBB BBG BGB BGG GBB GBG GGB GGG

Three outcomes (BBG, BGB, GBB) have one girl.

The probability of having one of three children being girls is 3/8.

Ex.) 5 – Classical Probability

If two dice are rolled one time, find the probability of getting a sum of 3 or 9.

Die 1	Die 2					
	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

$$P(\text{sum of } 3 \text{ or } 9) = \frac{2+4}{36} = \frac{6}{36} = \boxed{\frac{1}{6}}$$

Classical Probability

- What is the probability of choosing a king from a standard deck of playing cards?
- What is the probability of choosing a green marble from a jar containing 5 red, 6 green and 4 blue marbles?

Classical Probability

- What is the probability of getting an odd number when rolling a single 6-sided die?
- What is the probability of landing on an odd number after spinning a spinner with 7 equal sectors numbered 1 through 7?
- What is the probability of choosing the letter i from the word probability?

Sample Spaces and Probability

The ***complement of an event E***, denoted by $\bar{E}$, is the set of outcomes in the sample space that are not included in the outcomes of event E.

$$\boldsymbol{P\left(E\right) = 1 - P\left(\bar{E}\right)}$$

Ex.) 6 - Complements

Find the complement of each event.

Event	Complement of the Event
Rolling a die and getting a 2	Getting a 1, 3, 4, 5, or 6
Selecting a number and getting an even	Getting an odd
Selecting a month and getting a month that begins with an A	Getting January, February, March, May, June, July, September, October, November, or December
Having a boy	Having a girl

Ex.) 7 - Complements

If the probability that a person has brown eyes is 1/7, find the probability that a person does not have brown eyes.

$$P(\text{not having brown eyes}) = 1 - P(\text{having brown eyes})$$

$$= 1 - \frac{1}{7} = \boxed{\frac{6}{7}}$$

Compliments

- What is the probability of choosing a bead that is not blue from a jar containing 5 red, 6 green, and 4 blue beads?

Sample Spaces and Probability

There are three basic interpretations of probability:

- **Classical probability**
- **Empirical probability**
- **Subjective probability**

Sample Spaces and Probability

Empirical probability relies on actual experience to determine the likelihood of outcomes.

$$\boxed{P(E) = \frac{f}{n} = \frac{\text{frequency of desired class}}{\text{Sum of all frequencies}}}$$

Ex.) 8 – Empirical Probability

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

a. A person has type A blood.

Type	Frequency
A	22
B	5
AB	2
O	21
	Total 50

$$P(\mathrm{A}) = \frac{f}{n}$$

$$= \frac{22}{50} = \boxed{\frac{11}{25}}$$

Ex.) 8 – Empirical Probability

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

b. A person has type AB or type O blood.

Type	**Frequency**
A	22
B	5
AB	2
O	21
	Total 50

$$P(\text{AB or O}) = \frac{2}{50} + \frac{21}{50}$$

$$= \boxed{\frac{23}{50}}$$

Ex.) 8 – Empirical Probability

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

c. A person has neither type B nor type O blood.

Type	Frequency
A	22
B	~~5~~
AB	2
O	~~21~~
	Total 50

$$P(\text{neither B nor O}) = \frac{22}{50} + \frac{2}{50}$$

$$= \frac{24}{50} = \boxed{\frac{12}{25}}$$

Ex.) 8 – Empirical Probability

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

d. A person does not have type O blood.

Type	Frequency
A	22
B	5
AB	2
O	21
	Total 50

$$P(\text{not O}) = 1 - P(\text{O})$$

$$= 1 - \frac{21}{50} = \boxed{\frac{29}{50}}$$

Empirical Probability

“Is my class watching too much television on school nights?” This was a question that Mrs. Gordon wondered with respect to her 7th grade class. She did a quick poll in class and found the following:

Hours	Number
0	2
1	3
2	2
3	0
4	3
5	2
6	1

- What percentage of the class is not watching television on school nights?
- What percentage of the class is watching at most 2 hours of television on school nights?
- What percentage is watching at least 4 hours of television on school nights?

Sample Spaces and Probability

There are three basic interpretations of probability:

- **Classical probability**
- **Empirical probability**
- **Subjective probability**

Sample Spaces and Probability

Subjective probability uses a probability value based on an educated guess or estimate, employing opinions and inexact information.

Examples: weather forecasting, predicting outcomes of sporting events

Class Activity – Sample Spaces and Probability

Find the sample space for selecting 2 colors at random from the following: red, blue, green (colors can be repeated). Hint: First draw a tree diagram.

What is the probability of getting 2 reds?

What is the probability that you will not get 2 reds?

Someone comes along and says they have drawn 2 reds from a bag 5 times in a row. Does this mean that your next selection is more likely to be 2 reds?

Addition Rules for Probability

- Two events are **mutually exclusive events** if they cannot occur at the same time (i.e., they have no outcomes in common)

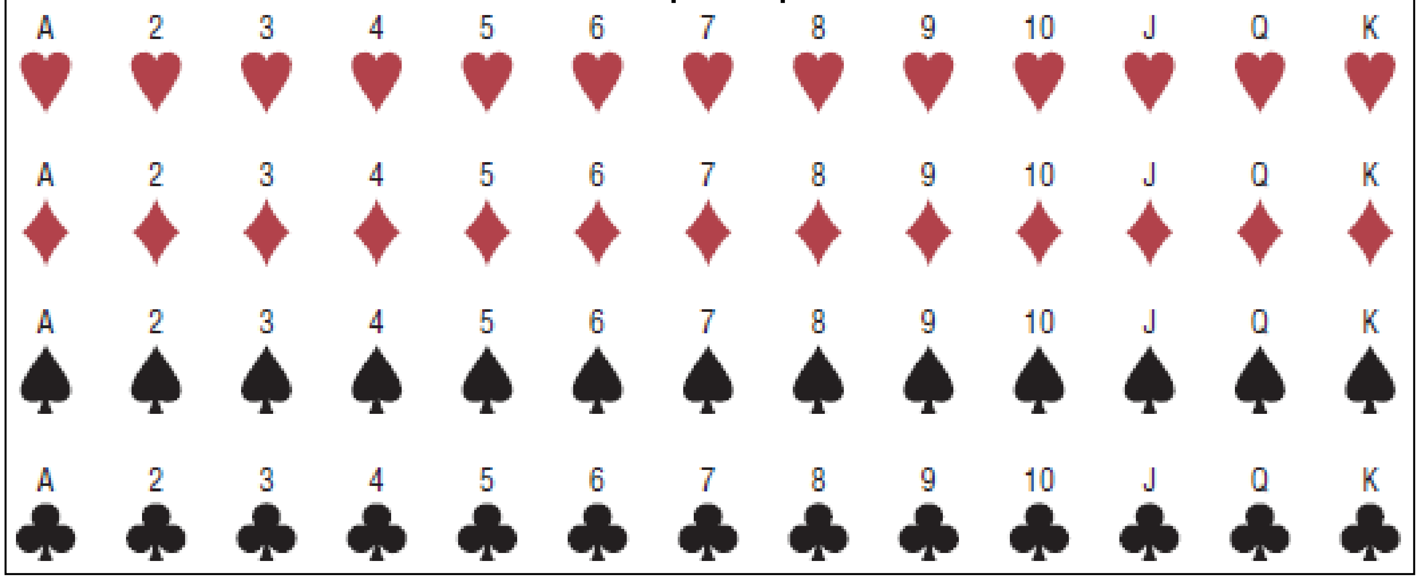
$$\boxed{\begin{array}{c}\text{Addition Rules}\\ P(A\,\text{or}\,B)=P(A)+P(B)\quad\text{Mutually Exclusive}\\ P(A\,\text{or}\,B)=P(A)+P(B)-P(A\text{ and }B)\quad\text{Not M. E.}\end{array}}$$

Ex.) 9 – Mutually Exclusive Events

Determine which events are mutually exclusive and which are not, when a card is drawn from a standard deck.

a. Getting an 8 and getting a king

No cards have both an 8 and a king → $\boxed{\text{Mutually Exclusive}}$



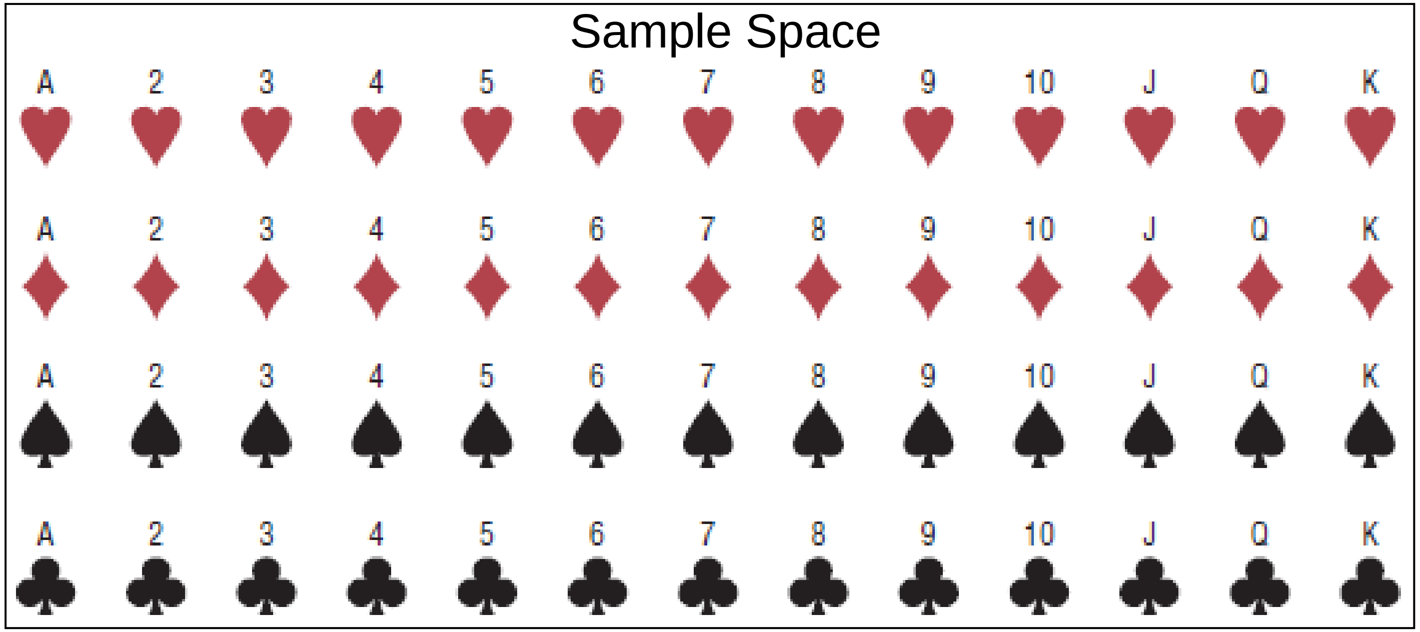
Sample Space

Ex.) 9 – Mutually Exclusive Events

Determine which events are mutually exclusive and which are not, when a card is drawn from a standard deck.

b. Getting a king and getting a face card

A king is a face card → $\boxed{\text{Not Mutually Exclusive}}$



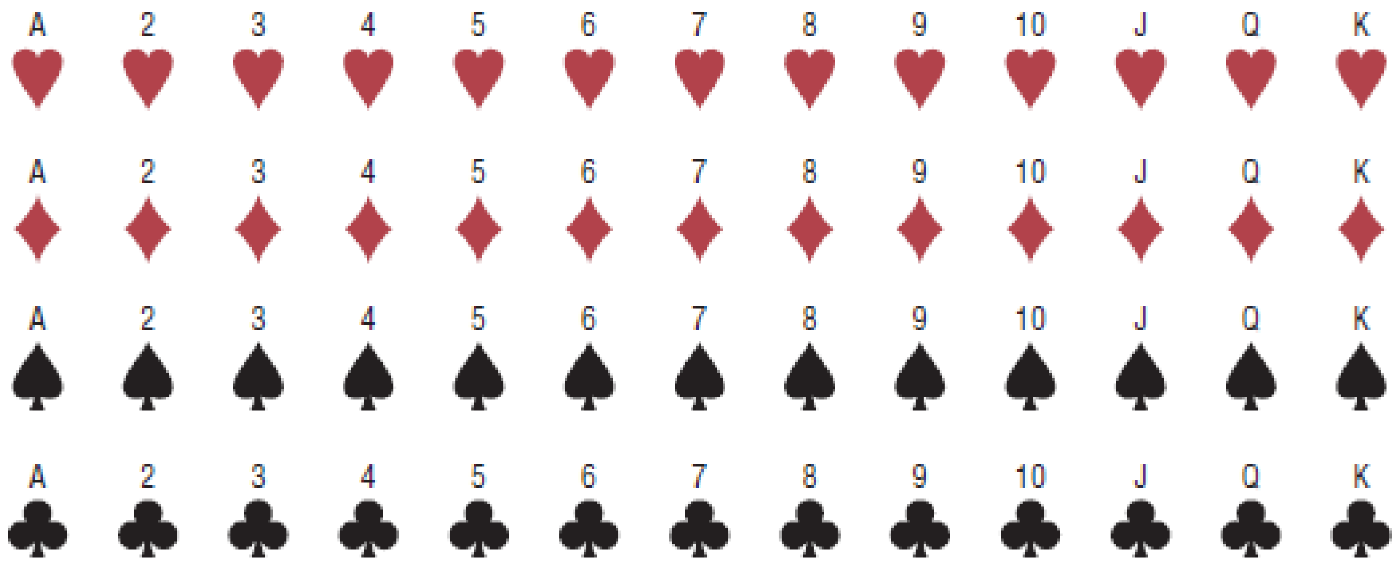
Ex.) 9 – Mutually Exclusive Events

Determine which events are mutually exclusive and which are not, when a card is drawn from a standard deck.

c. Getting a face card and getting an ace

An ace is not a face card → Mutually Exclusive

Sample Space



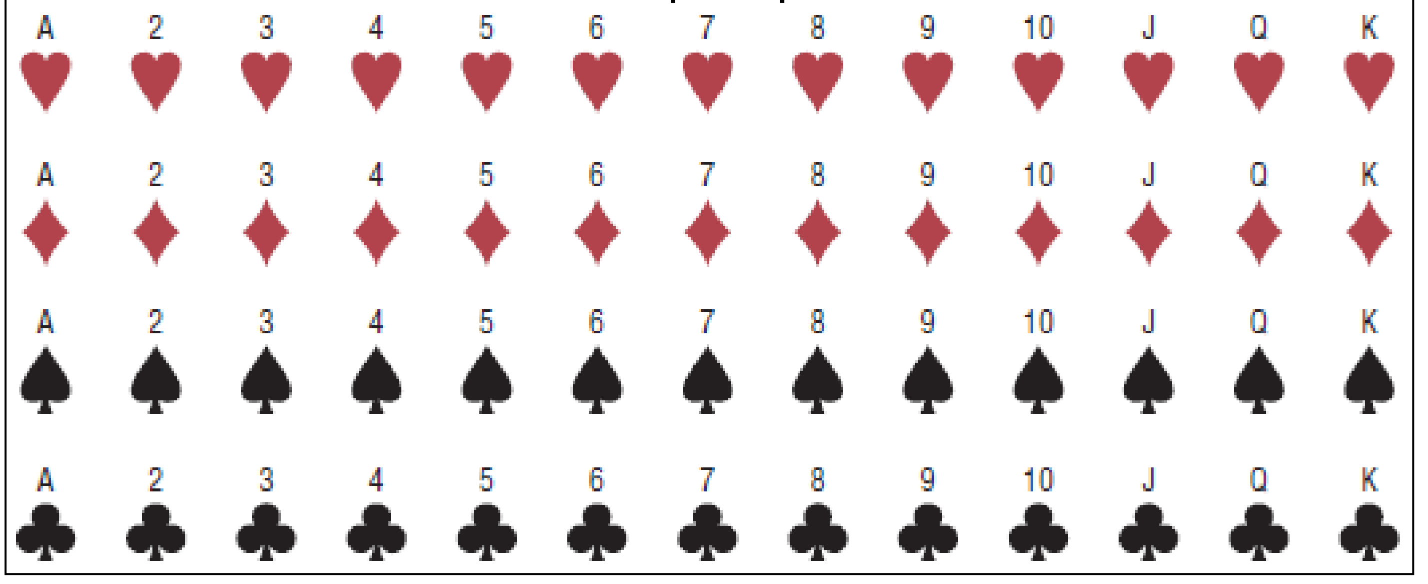
Ex.) 9 – Mutually Exclusive Events

Determine which events are mutually exclusive and which are not, when a card is drawn from a standard deck.

d. Getting an ace and getting a heart

The ace of hearts is a card → $\boxed{\text{Not Mutually Exclusive}}$

Sample Space



Ex.) 10 – Mutually Exclusive Events

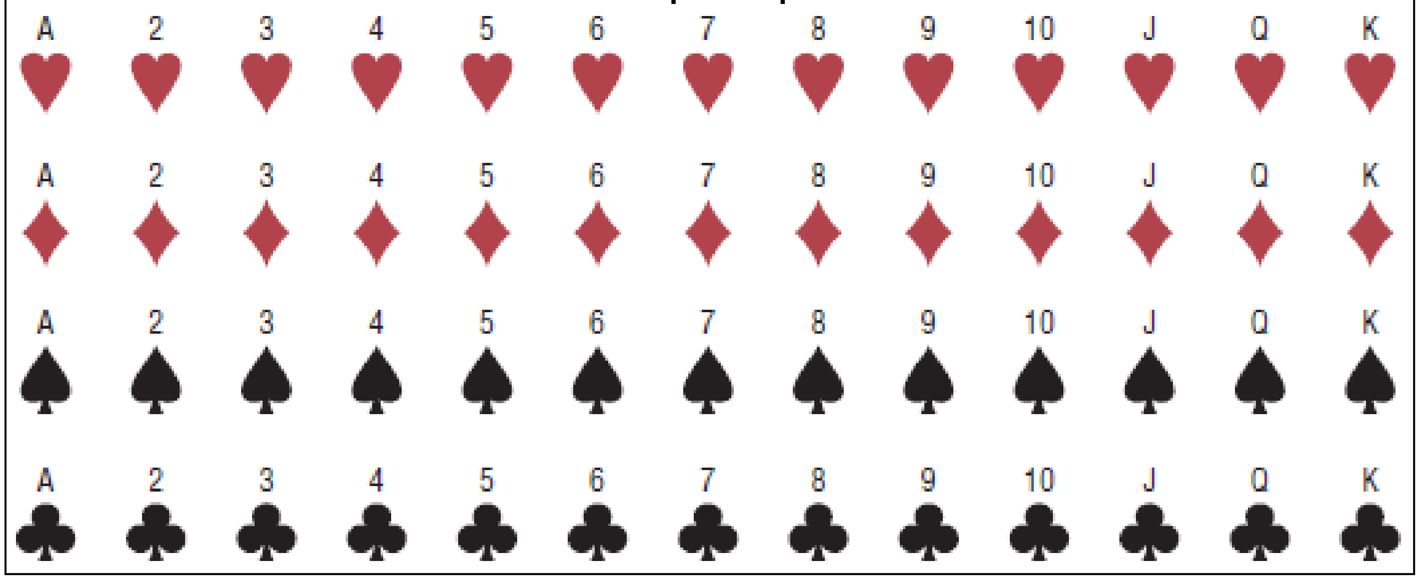
A single card is drawn from a deck. Find the probability of selecting the following:

a. A club or a diamond

Mutually Exclusive Events

$$P(\text{club or diamond}) = P(\text{club}) + P(\text{diamond}) = \frac{13}{52} + \frac{13}{52} = \frac{26}{52} = \boxed{\frac{13}{26}}$$

Sample Space



Ex.) 10 – Mutually Exclusive Events

A single card is drawn from a deck. Find the probability of selecting the following:

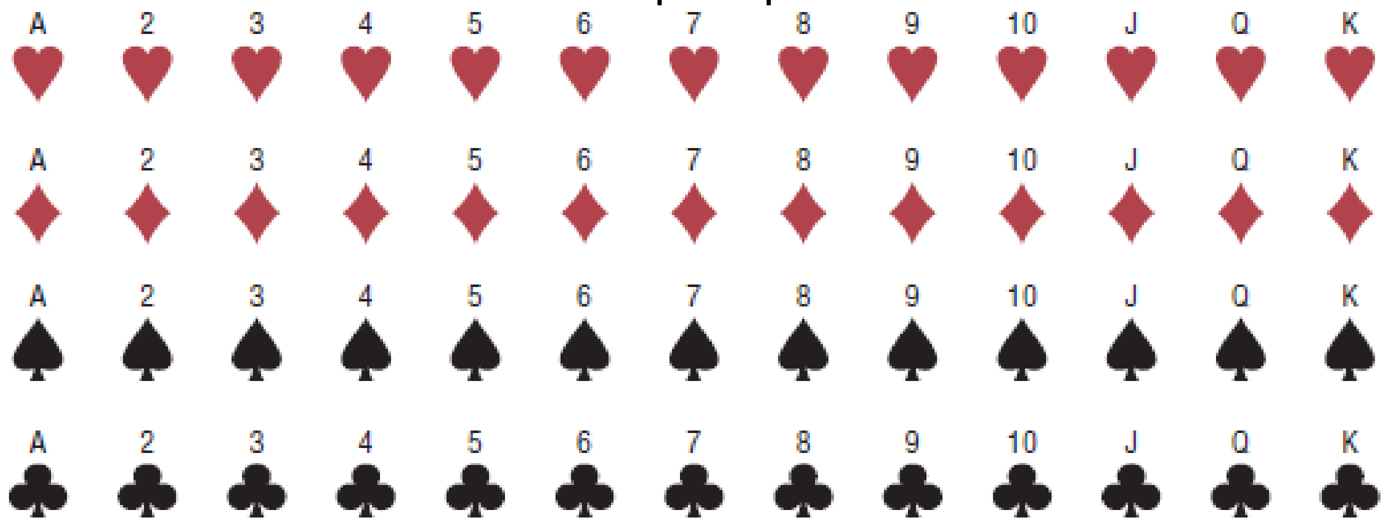
a. A 4 or a diamond

Not Mutually Exclusive Events

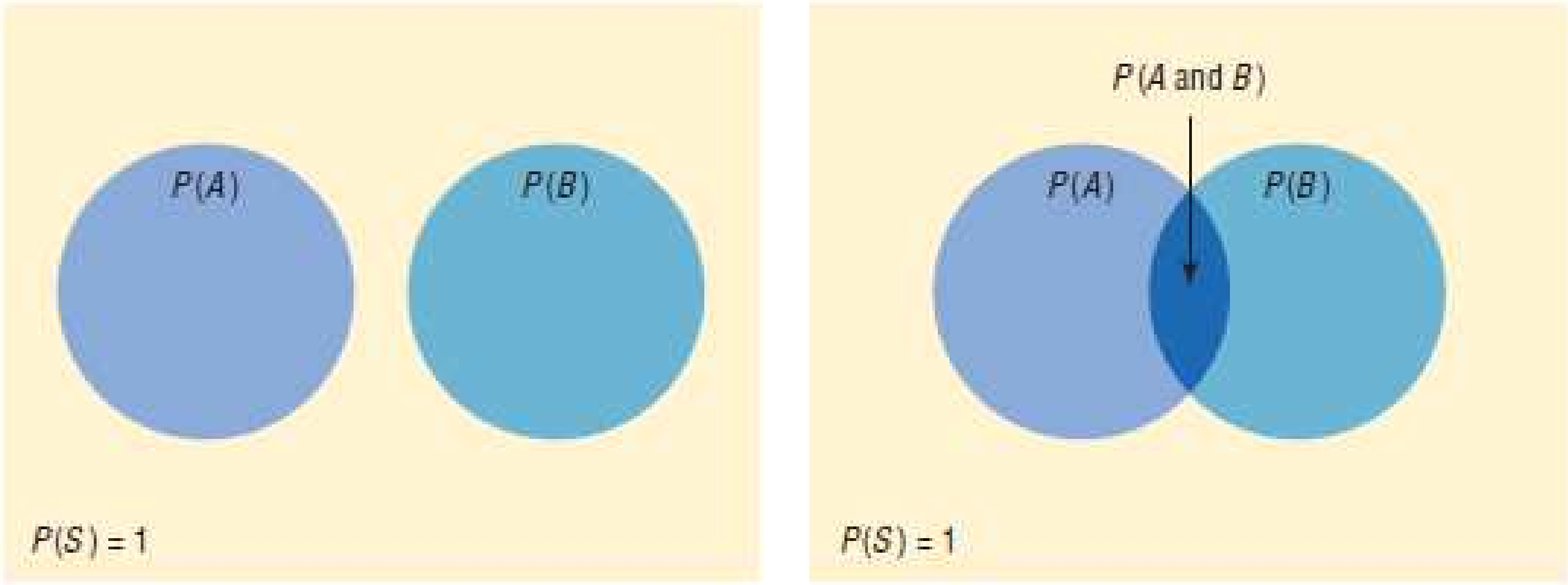
$$P(4\text{ or diamond}) = P(4) + P(\text{diamond}) - P(\text{diamond } 4)$$

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \boxed{\frac{4}{13}}$$

Sample Space



Mutually Exclusive Events – Venn diagram



(a) Mutually exclusive events
$P(A \text{ or } B) = P(A) + P(B)$

(b) Nonmutually exclusive events
$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Mutually Exclusive Events

- In the tv watching example with the following table, what is the probability that a student watches 2 or 4 hours of tv? Are these mutually exclusive events?

Hours	Number
0	2
1	3
2	2
3	0
4	3
5	2
6	1

Mutually Exclusive Events

- You want to select two single digit numbers between 0 and 4. What is the probability that the two numbers add up to 2 or multiply to 2? Are these mutually exclusive events?

Multiplication Rules

- Two events *A* and *B* are **independent events** if the fact that *A* occurs does not affect the probability of *B* occurring.

Multiplication Rules

$$P(A \text{ and } B) = P(A) \cdot P(B) \quad \text{Independent}$$

$$P(A \text{ and } B) = P(A) \cdot P(B|A) \quad \text{Dependent}$$

Ex.) 11 – Independent Events

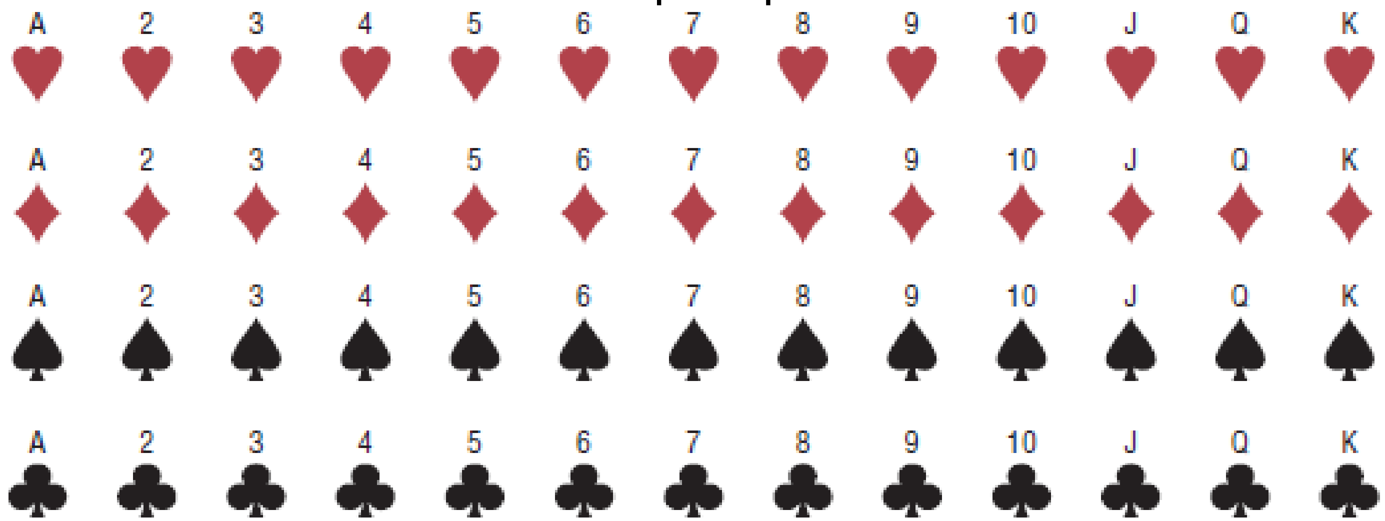
Two cards are drawn from a deck *with replacement*. Find the probability that both cards are spades.

Independent events

$$P(\text{card 1 spade and card 2 spade}) = P(\text{card 1 spade}) \cdot P(\text{card 2 spade})$$

$$= \frac{13}{52} \cdot \frac{13}{52} = \frac{169}{2704} = \boxed{\frac{1}{16}}$$

Sample Space



Ex.) 12 – Independent Events

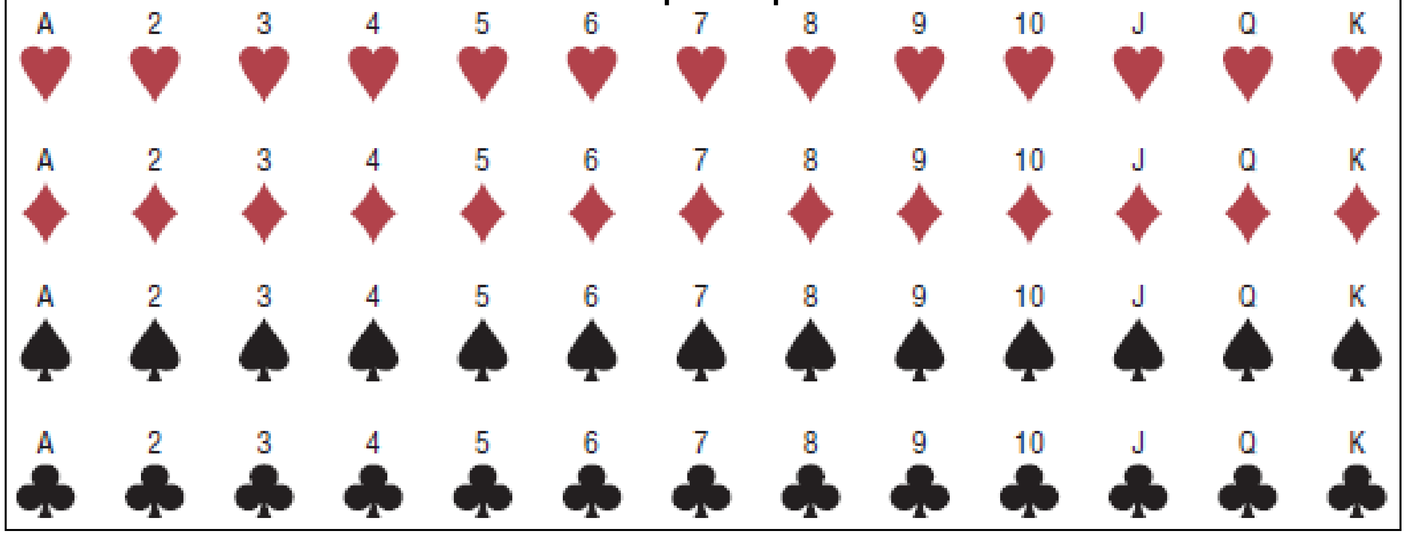
Three cards are drawn from a deck *with replacement*. Find the probability that all are jacks.

Independent Events

$$P(\text{all are jacks}) = P(\text{card 1 jack}) \cdot P(\text{card 2 jack}) \cdot P(\text{card 3 jack})$$

$$= \frac{4}{52} \cdot \frac{4}{52} \cdot \frac{4}{52} = \frac{64}{140608} = \boxed{\frac{1}{2197}}$$

Sample Space



Ex.) 13 – Independent Events

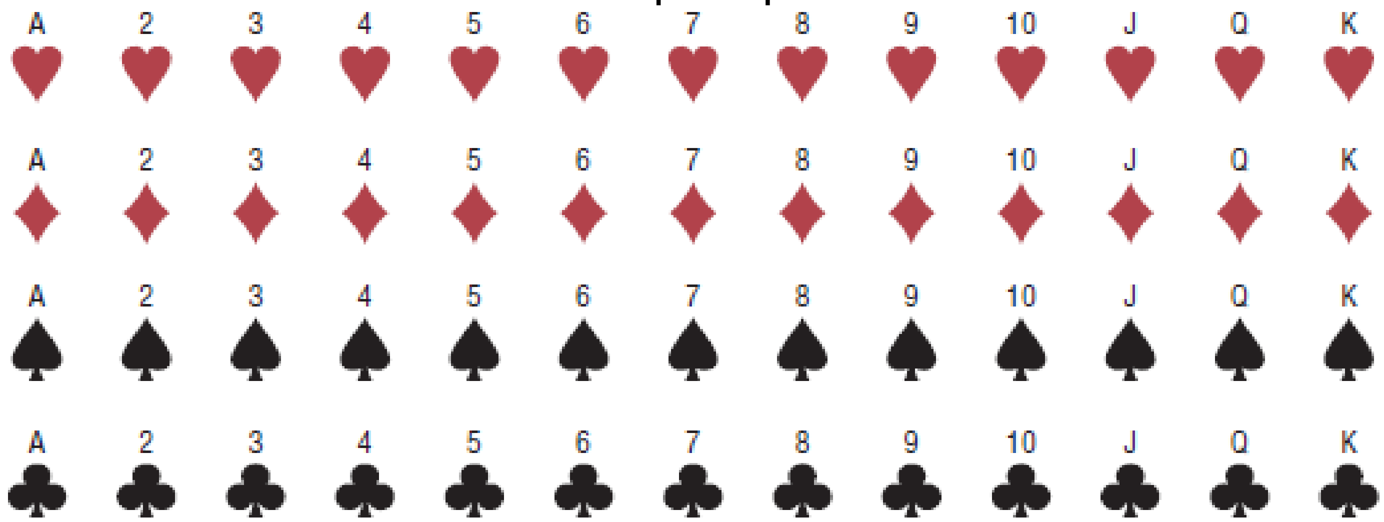
Two cards are drawn from a deck *without replacement*. Find the probability that both cards are spades.

Dependent events

$$P(\text{both cards are spades}) = P(\text{card 1 spade}) \cdot P(\text{card 2 spade} \mid \text{card 1 spade})$$

$$= \frac{13}{52} \cdot \frac{12}{51} = \frac{156}{2652} = \boxed{\frac{1}{17}}$$

Sample Space



Ex.) 14 – Independent Events

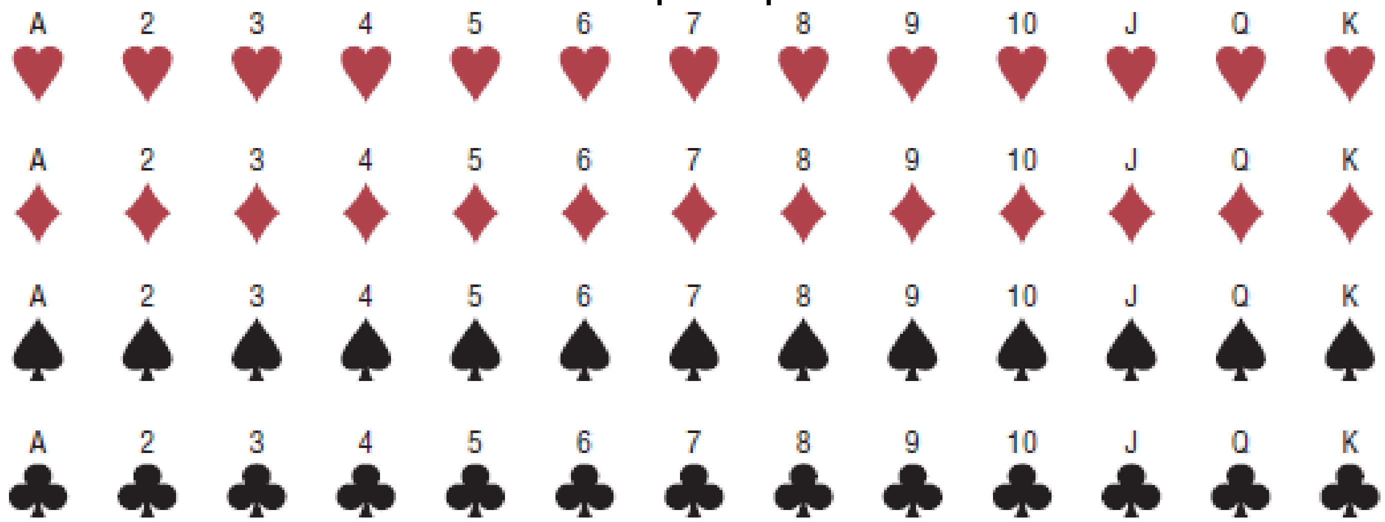
Two cards are drawn from a deck *without replacement*. Find the probability that both cards are the same suit.

Dependent events

$$P(\text{both cards are same suit}) = P(\text{card 1 is a suit}) \cdot P(\text{card 2 same suit as card 1})$$

$$= \frac{4}{4} \cdot \frac{12}{51} = \frac{12}{51} = \boxed{\frac{4}{17}}$$

Sample Space



Independent events

- What is the probability of choosing a red bead and then a blue bead from a jar containing 5 red, 6 green, and 4 blue beads if we select 2 beads from the jar with replacement? Are these independent events?

Independent events

- What is the probability of choosing a red bead and then a blue bead from a jar containing 5 red, 6 green, and 4 blue beads if we select 2 beads from the jar without replacement? Are these independent events?

Conditional Probability

- **Conditional probability** is the probability that the second event *B* occurs given that the first event *A* has occurred.

$$\boxed{\begin{gathered}\text{Conditional Probability}\\ P(B|A)=\frac{P(A\,\text{and}\,B)}{P(A)}\end{gathered}}$$

Ex.) 15 – Conditional Probability

In Rolling Acres Housing Development, 42% of the houses have a deck and a garage; 60% have a deck. Find the probability that a home has a garage, given that it has a deck.

G = has a garage, D = has a deck

$$P(\mathrm{G} \mid \mathrm{D}) = \frac{P(\mathrm{D\ and\ G})}{P(\mathrm{D})} = \frac{0.42}{0.60} = \boxed{0.70}$$

Ex.) 16 – Conditional Probability

In addition to being grouped into four types, human blood is grouped by its Rh factor. Consider the following distribution for Americans.

	O	A	B	AB
Rh+	37%	34%	10%	4%
Rh−	6%	6%	2%	1%

Ex.) 16 – Conditional Probability

a. Find the probability that a person has type O blood given that they are Rh+.

	O	A	B	AB
Rh+	37%	34%	10%	4%
Rh−	6%	6%	2%	1%

$$P(\mathrm{O} \mid +) = \frac{P(+\ \text{and}\ \mathrm{O})}{P(+)} = \frac{0.37}{0.85} = \boxed{0.435}$$

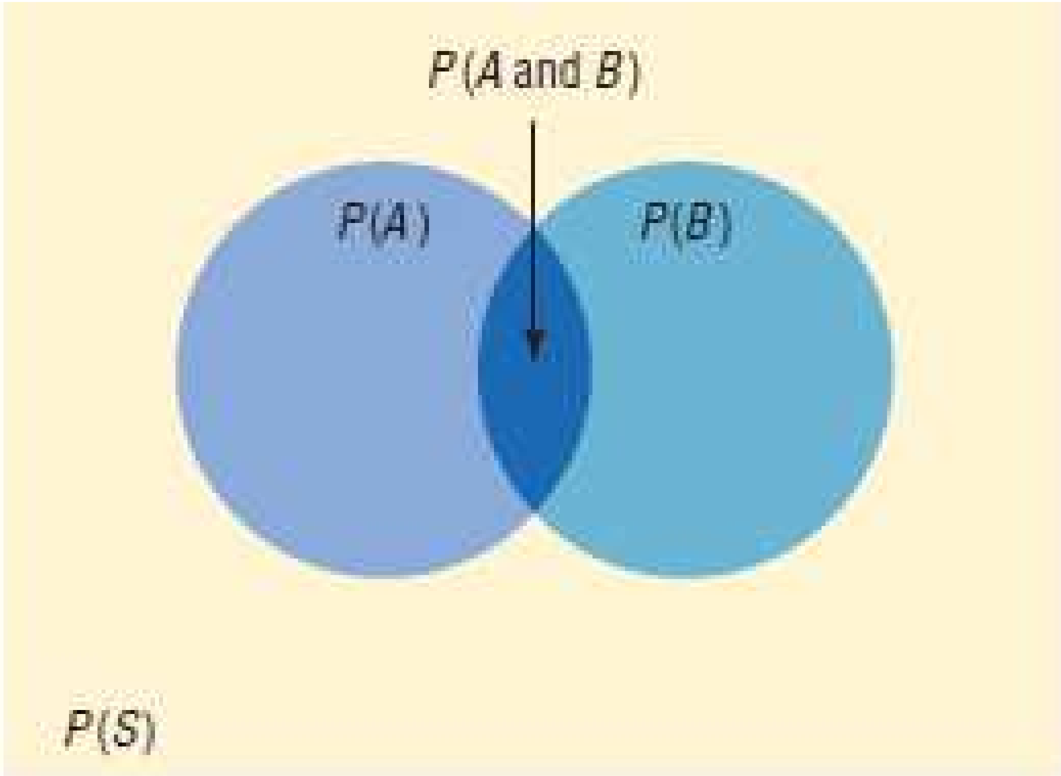
Ex.) 16 – Conditional Probability

b. Find the probability that a person is Rh- given that they have type B blood.

	O	A	B	AB
Rh+	37%	34%	10%	4%
Rh−	6%	6%	2%	1%

$$P(- \mid \mathrm{B}) = \frac{P(\mathrm{B\ and\ -})}{P(\mathrm{B})} = \frac{0.02}{0.12} = \boxed{0.167}$$

Conditional Probability – Venn Diagram



$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Conditional Probability

- Given the following table, what is probability the person selected is a child, given that the person watches TV > 10 hours/week?

	Adult	Child	Total
Watches TV > 10 hours/week	75	150	225
Watches TV <= 10 hours/week	125	50	175
Total	200	200	400

Class Activity

- Given the following table, what is probability the person selected watches TV <= 10 hours/week given that the person is an Adult?

	Adult	Child	Total
Watches TV > 10 hours/week	75	150	225
Watches TV <= 10 hours/week	125	50	175
Total	200	200	400

Ex.) 17 – Conditional Probability

Suppose the probability that a woman who purchases an early pregnancy test is actually pregnant is 70%. The probability that the test is positive when the woman is actually pregnant is 95%. The probability that the test is positive when the woman is not pregnant is 1%.

a) Construct a probability tree for this problem and label the probabilities along the branches.
b) What is the probability that a woman is actually pregnant given that she had a positive test?

Ex.) 17 – Conditional Probability

We are given:

P(pregnant) = .70 P(+ | pregnant) = .95 P(+ | not pregnant) = .01

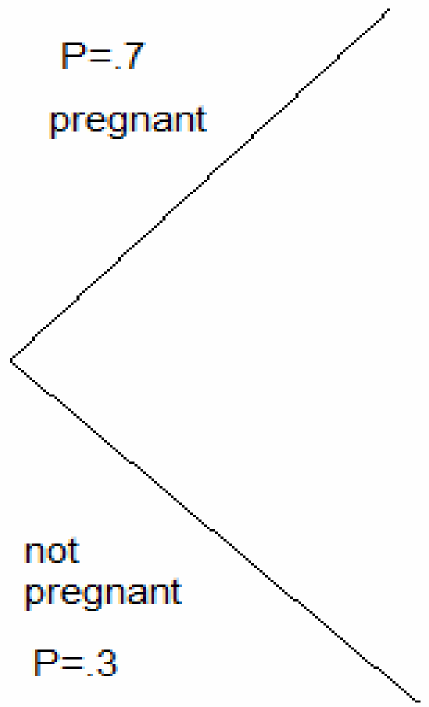
From this we can find the other probabilities:

P(not pregnant) = .30 P(- | pregnant) = .05 P(- | not pregnant) = .99

We can draw a probability tree with this information.

Ex.) 17 – Conditional Probability

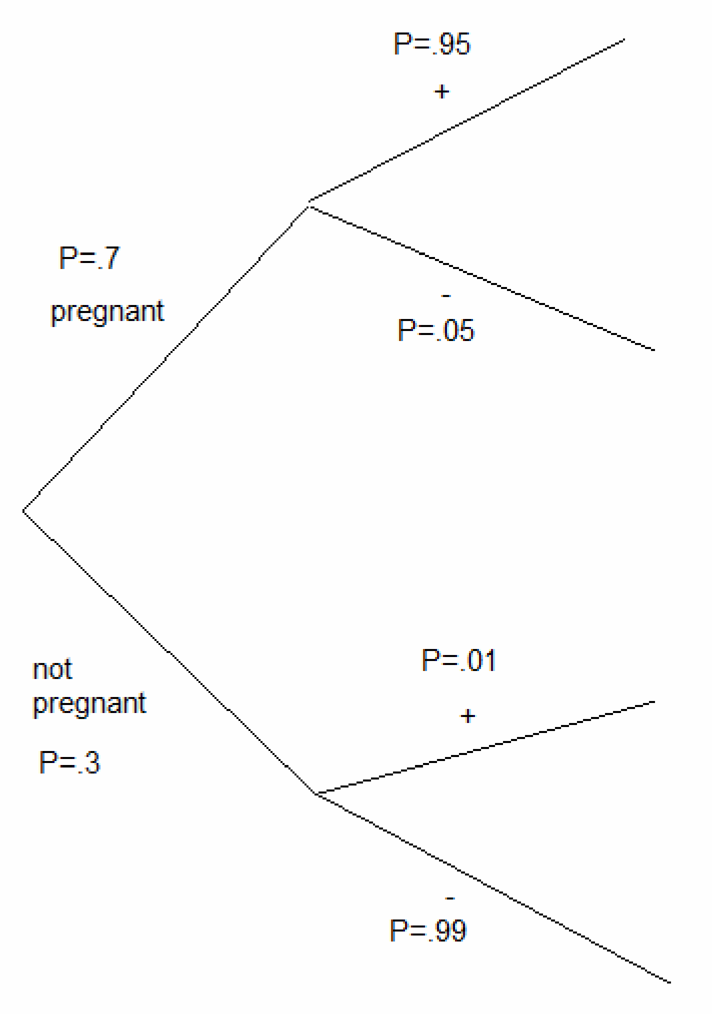
Start by making branches and label the probabilities. To begin, a woman taking a pregnancy test is either pregnant or not pregnant.



What we Know:
P(pregnant) = .70
P(+ | pregnant) = .95
P(+ | not pregnant) = .01
P(not pregnant) = .30
P(- | pregnant) = .05
P(- | not pregnant) = .99

Ex.) 17 – Conditional Probability

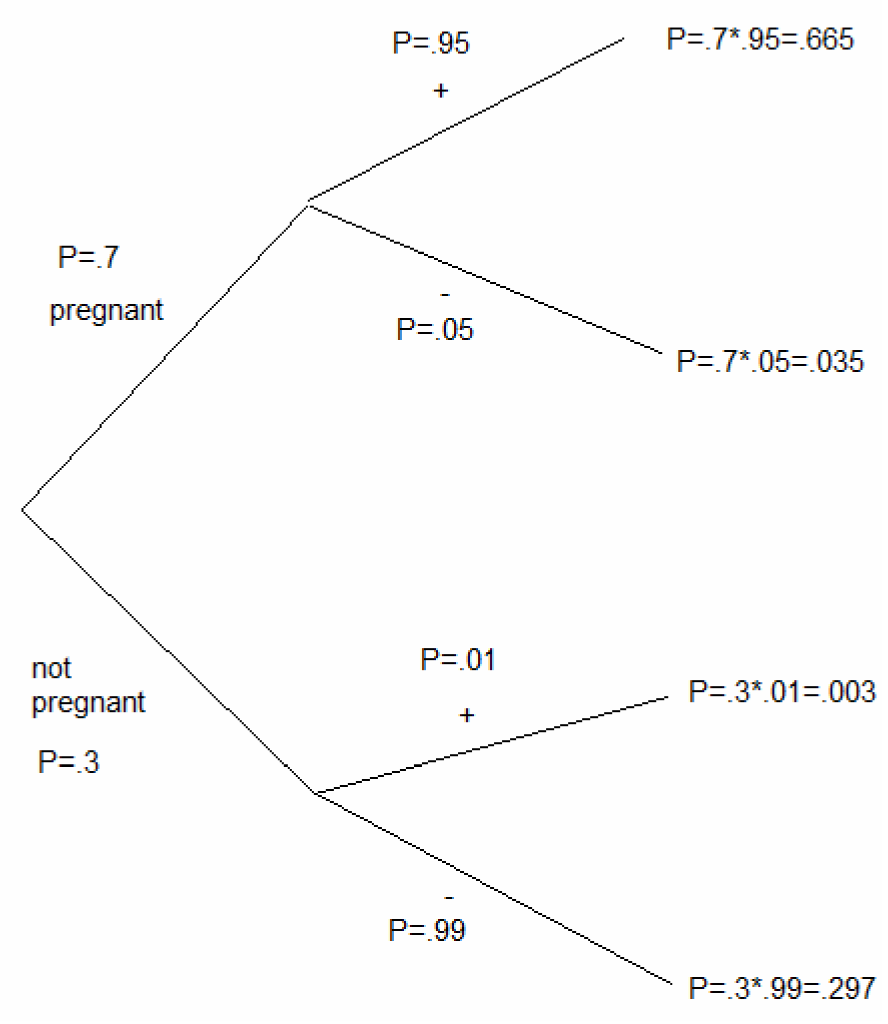
Continue the process by considering the next step – she takes a pregnancy test, and it is either positive or negative. Once again, label the probabilities



What we Know:
P(pregnant) = .70
P(+ | pregnant) = .95
P(+ | not pregnant) = .01
P(not pregnant) = .30
P(- | pregnant) = .05
P(- | not pregnant) = .99

Ex.) 17 – Conditional Probability

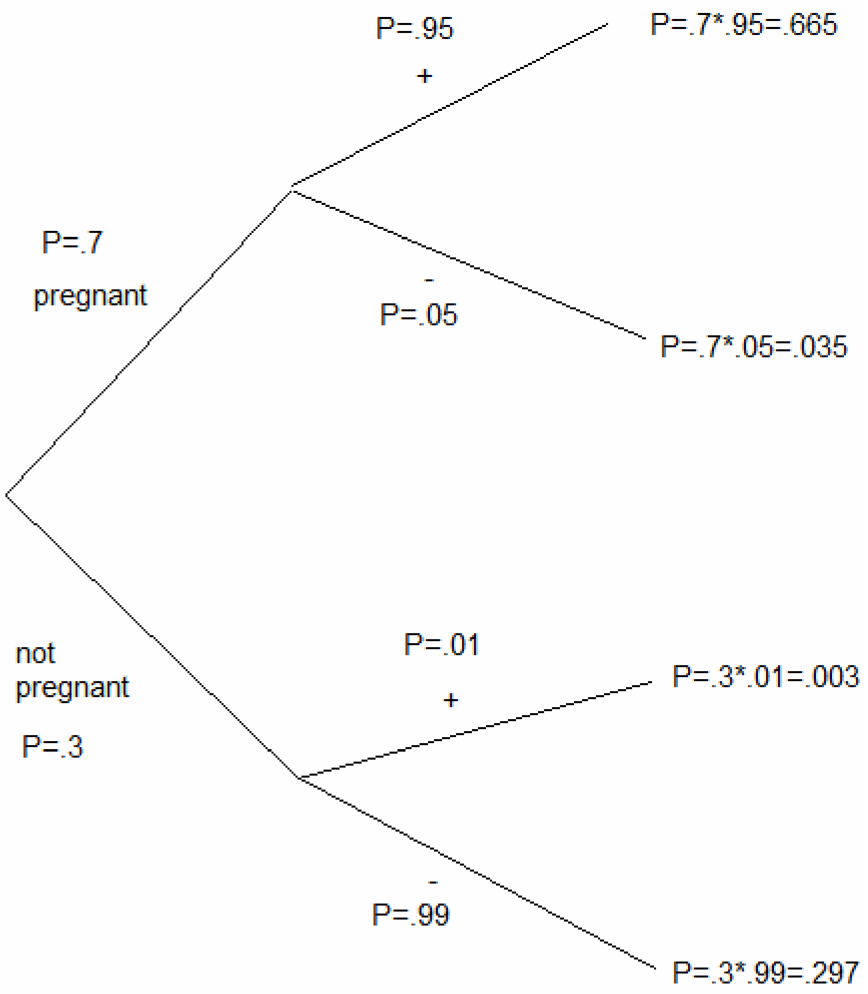
Find the probability of each path by multiplying across the branches.



Ex.) 17 – Conditional Probability

Now put this into a joint probability table.

	+	-	
Pregnant	.665	.035	.700
Not pregnant	.003	.297	.300
	.668	.332	1



Ex.) 17 – Conditional Probability

Now use the formula for conditional probability and the table values:

	+	-	
Pregnant	.665	.035	.700
Not pregnant	.003	.297	.300
	.668	.332	1

$$P(B \mid A) = \frac{P(A \& B)}{P(A)}$$

$$P(pregnant \mid +) = \frac{P(+ \& \; pregnant)}{P(+)} = \frac{.665}{.668} = .996$$

Class Activity – Mutually Exclusive, Independent Events, & Conditional Probability

Are the following events mutually exclusive, independent, or neither?

a. One student is selected at random from a student body: the person selected is “male”, the person selected is “older than 21”.

b. Two dice are rolled: the total showing is “less than 7”, the total showing is “more than 9”.

c. Rolling a pair of dice and observing a “2” on one of the dice and having a “total of 10”.

c. Rolling a pair of dice and observing a “1” on the first die and a “1” on the second die.

Find the probability of selecting a male student or selecting a student older than 21 if the student body is 1000 students and there are 400 male students, 300 students over 21, and 150 male students that are over 21.

Find the probability that you roll a product of 6 given that one of the dice is a 6.

Properties of Probability Numbers

- Whether the probability is *empirical*, *theoretical*, or *subjective*, the following properties must hold.

Property 1

In words: "A probability is always a numerical value between zero and one."

In algebra: $0 \leq \text{each } P(A) \leq 1$ or $0 \leq \text{each } P'(A) \leq 1$

- **Notes about Property 1:**
- The probability is 0 if the event cannot occur.
- The probability is 1 if the event occurs every time.

Properties of Probability Numbers

- Otherwise, the probability is a fractional number between 0 and 1.

Property 2

In words: "The sum of the probabilities for all outcomes of an experiment is equal to exactly one."

In algebra: $\sum_{\text{all outcomes}} P(A) = 1$ or $\sum_{\text{all outcomes}} P'(A) = 1$

- **Note about Property 2:** The list of "all outcomes" must be a nonoverlapping set of events that includes all the possibilities (**all-inclusive**).

Properties of Probability Numbers

- **Notes about probability numbers:**
- Probability represents a relative frequency, whether from a sample space or a sample.
- $P(A)$ is the ratio of the number of times an event can be expected to occur divided by the number of possibilities.

  $P'(A)$ is the ratio of the number of times an event did occur divided by the number of data.
- The numerator of the probability ratio must be a positive number or zero.

Properties of Probability Numbers

- The denominator of the probability ratio must be a positive number (greater than zero).

- As a result of the notes above, the probability of an event, whether it be empirical, theoretical, or subjective, will always be a numerical value between zero and one, inclusive.

- The rules for probability are the same for all three types of probability: empirical, theoretical, and subjective.

Class Activity – Probability

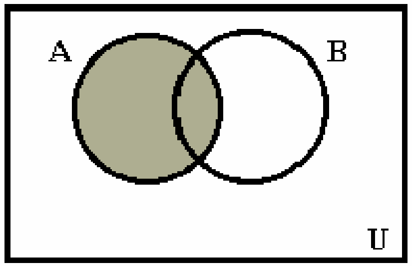
Let's look at the sample space for rolling a pair of dice.

Find the probability of getting:
a. A sum of 10 or 11
b. A sum greater than 10
c. A sum less than 4 or greater than 10
d. A sum that is divisible by 3
e. A sum of 9
f. A sum less than 13

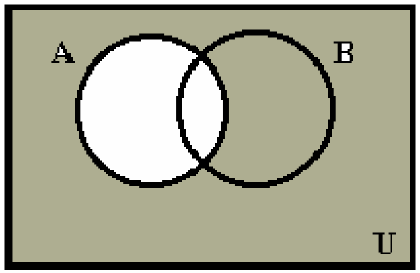
If you had to bet on a particular number for the sum of the 2 dice rolled, what would it be and why?

Venn Diagram Review

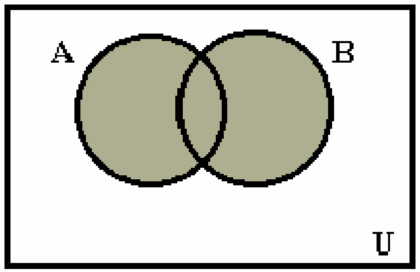
Event A



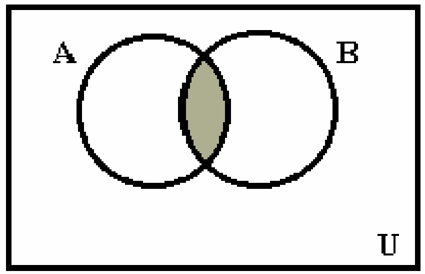
Complement of A
'not A'



Union of A and B
'A or B'



Intersection of A and B
'A and B'



Practice

Use a Venn diagram to find an equivalent statement for each of the following:

a) $P(\overline{A} \text{ and } \overline{B})$.

b) $P(\overline{A} \text{ or } B)$.