Chapter 5

Discrete Probability Distributions

Chapter 5 Overview

- 5-1 Random Variables
- 5-2 Probability Distributions of a Discrete Random Variable
- 5-3 The Binomial Probability Distribution

Review: Discrete & Continuous

Determine if the following are discrete or continuous random variables:

- The speed of a race car in mph.
- The number of cups of coffee that Mrs. Lowery drinks each day.
- The number of people that play the SC Lottery each day.
- The weight of a rhinoceros.
- The time it takes to complete Mrs. Lowery’s midterm.
- The number of math majors at USC.
- The blood pressures of patients at Lexington Medical Center.

Introduction

A. Roll a pair of dice to complete the table below.

Die 1	Die 2	Sum		Die 1	Die 2	Sum
⚀	⚀	2		⚃	⚀	
⚀	⚁	3		⚃	⚁	
⚀	⚂	4		⚃	⚂	
⚀	⚃	5		⚃	⚃	
⚀	⚄	6		⚃	⚄	
⚀	⚅	7		⚃	⚅	
⚁	⚀	3		⚄	⚀	
⚁	⚁	4		⚄	⚁	
⚁	⚂	5		⚄	⚂	
⚁	⚃			⚄	⚃	
⚁	⚄			⚄	⚄	
⚁	⚅			⚄	⚅	
⚂	⚀			⚅	⚀	
⚂	⚁			⚅	⚁	
⚂	⚂			⚅	⚂	
⚂	⚃			⚅	⚃	
⚂	⚄			⚅	⚄	
⚂	⚅			⚅	⚅	

Introduction

B. Use the table above to answer the questions below.

1. When you roll a pair of dice, how many outcomes are there? ____________

2. How many ways can you get a sum of 1? ______
3. How many ways can you get a sum of 2? ______
4. How many ways can you get a sum of 3? ______
5. How many ways can you get a sum of 4? ______
6. How many ways can you get a sum of 5? ______
7. How many ways can you get a sum of 6? ______
8. How many ways are there to get a sum of 7? ______
9. How many ways are there to get a sum of 8? ______
10. How many ways are there to get a sum of 9? ______
11. How many ways are there to get a sum of 10? ______
12. How many ways are there to get a sum of 11? ______
13. How many ways are there to get a sum of 12? ______

C. Complete the table below.

Sum of dice	2	3	4	5	6	7	8	9	10	11	12
# outcomes											

Introduction

- We just created what is called a *Discrete Probability Distribution*.
- Let’s take a closer look at these discrete probability distributions.

5-1 Random Variables

- A **random variable** is a variable whose values are determined by chance.
- A **discrete probability distribution** consists of the values a random variable can assume and the corresponding probabilities of the values.
- The sum of the **probabilities** of all events in a sample space add up to 1. Each probability is between 0 and 1, inclusively.

5-2 Probability Distributions of a Discrete Random Variable

Ex.) 1 – Probability Distributions

Construct a probability distribution for rolling a single die.

Outcome X	1	2	3	4	5	6
Probability $P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

5-2 Discrete Probability Distributions

You flip four coins. Let X, the random variable, be the number of heads on all four coins.

- List the sample space for the experiment.
- What are the possible values for x?
- Is the random variable, x, continuous or discrete?
- Construct a probability distribution for this experiment.

- Construct a histogram for the probability distribution.

Class Activity

Construct a probability distribution for the data and draw a histogram for the following:

- The probabilities that a patient will have 0, 1, 2, or 3 medical tests performed on entering a hospital are 6/15, 5/15, 3/15, and 1/15 respectively.

Class Activity

Construct a probability distribution for the data and draw a histogram for the following:

- A box contains 3 \$1 bills, 2 \$5 bills, 1 \$10 bill, and 1 \$20 bill.

Probability Distributions

Two Requirements for a Probability Distribution

1. The sum of the probabilities of all the events in the sample space must equal 1; that is, $\Sigma P(X) = 1$.
2. The probability of each event in the sample space must be between or equal to 0 and 1. That is, $0 \leq P(X) \leq 1$.

Ex.) 2 – Probability Distributions

Determine whether the following is a discrete probability distribution:

X	3	6	8	12
$P(X)$	0.3	0.5	0.7	−0.8

1. Do all events in the sample space sum to 1?

$$0.3+0.5+0.7+(-0.8)=0.7\neq 1$$

$$\Rightarrow \text{Not a probability distribution}$$

2. Is each probability between 0 and 1, inclusively?

   No, -0.8 is not between 0 & 1

   → Not a Probability Distribution

Discrete Probability Distributions

Determine if the following is a probability distribution (if not, state why).

X	3	6	9	12	15
P(X)	4/9	2/9	1/9	1/9	1/9

Class Activity

Determine if the following is a probability distribution (if not, state why).

X	20	30	40	50
P(X)	1.1	0.2	0.9	0.3

Discrete Probability Distributions

- Below is a probability distribution for the number of math failures of BC students.

X	0	1	2	3	4
P(X)	.41	.38		.08	.02

- $P(X = 2)$
- $P(X < 2)$
- $P(X \le 2)$
- $P(X \le 1)$
- $P(X > 2)$
- $P(X = 3 \text{ or } X = 4)$

Mean and Variance of a Discrete Probability Distribution

Probability distributions may be used to represent theoretical populations, the counterpart to samples.

We use **population parameters** (mean, variance, and standard deviation) to describe these probability distributions just as we use **sample statistics** to describe samples.

Notes

1. $\bar{x}$ is the mean of the sample.
2. s^2 and s are the variance and standard deviation of the sample, respectively.

Mean and Variance of a Discrete Probability Distribution

3. $\bar{x}$, s^2, and s are called *sample statistics*.

4. μ (lowercase Greek letter mu) is the mean of the population.

5. σ^2 (sigma squared) is the variance of the population.

6. σ (lowercase Greek letter sigma) is the standard deviation of the population.

Mean and Variance of a Discrete Probability Distribution

7. μ, σ^2 and σ are called *population parameters*. (A parameter is a constant; σ, σ^2, and σ are typically unknown values in real statistics problems. About the only time they are known is in a textbook problem setting for the purposes of learning and understanding.)

Mean, Variance, Standard Deviation, and Expectation

$$\text{MEAN:}\ \ \mu = \sum X \cdot P(X)$$

VARIANCE:

$$\sigma^2 = \sum \left[X^2 \cdot P(X) \right] - \mu^2$$

Mean, Variance, Standard Deviation, and Expectation

Rounding Rule

The mean, variance, and standard deviation should be rounded to one more decimal place than the outcome *X.*

When fractions are used, they should be reduced to lowest terms.

Ex.) 3 – Mean, Variance, Standard Deviation, and Expectation

Find the mean of the number that appears when a card is drawn from a standard deck.

Outcome X	1	2	3	4	5	6	7	8	9	10	11	12	13
Probability $P(X)$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$

$$\mu = \sum X \cdot P(X)$$

Ex.) 3 – Mean, Variance, Standard Deviation, and Expectation

X	P(X)	X*P(X)
1	1/13	1/13
2	1/13	2/13
3	1/13	3/13
4	1/13	4/13
5	1/13	5/13
6	1/13	6/13
7	1/13	7/13
8	1/13	8/13
9	1/13	9/13
10	1/13	10/13
11	1/13	11/13
12	1/13	12/13
13	1/13	13/13
Sum of X*P(X)		**91/13**

So our expectation, μ, is the Sum of X*P(X) which is:

$$\mu = \frac{91}{13} = 7$$

Ex.) 4 – Mean, Variance, Standard Deviation, and Expectation

A bank vice president feels that each savings account holder has on average 3 credit cards. The following represents the distribution of the number of credit cards owned. Find the mean number of credit cards owned. Is the vice president correct?

Number of cards X	0	1	2	3	4
Probability $P(X)$	0.18	0.44	0.27	0.08	0.03

Class Activity

- The number of suits sold per day at Suit World is shown in the probability distribution below.

X	19	20	21	22	23
P(X)	0.2	0.2	0.3	0.2	0.1

- Find the mean of the distribution.

Ex.) 5 – Mean, Variance, Standard Deviation, and Expectation

Compute the variance and standard deviation for the probability distribution in Example 4.

Outcome X	1	2	3	4	5	6	7	8	9	10	11	12	13
Probability $P(X)$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$

$$\sigma^2 = \sum \left[X^2 \cdot P(X) \right] - \mu^2$$

Ex.) 5 – Mean, Variance, Standard Deviation, and Expectation

X	X^2	P(X)	X^2*P(X)
1	1	1/13	1/13
2	4	1/13	4/13
3	9	1/13	9/13
4	16	1/13	16/13
5	25	1/13	25/13
6	36	1/13	36/13
7	49	1/13	49/13
8	64	1/13	64/13
9	81	1/13	81/13
10	100	1/13	100/13
11	121	1/13	121/13
12	144	1/13	144/13
13	169	1/13	169/13
Sum of X^2*P(X)			**819/13 = 63**

So our variance, σ^2, is the Sum of X^2*P(X) – μ^2 which is:

$$\sigma^2 = 63 - 7^2 = 63 - 49 = 14$$

$$\sigma = \sqrt{14} = 3.74$$

Ex.) 6 – Mean, Variance, Standard Deviation, and Expectation

A bank vice president feels that each savings account holder has on average 3 credit cards. The following represents the distribution of the number of credit cards owned. Find the variance & standard deviation for the number of credit cards owned.

Number of cards X	0	1	2	3	4
Probability $P(X)$	0.18	0.44	0.27	0.08	0.03

Class Activity

- The number of suits sold per day at Suit World is shown in the probability distribution below.

X	19	20	21	22	23
P(X)	0.2	0.2	0.3	0.2	0.1

- Find the variance and standard deviation of the distribution.

Expectation

- The **expected value**, or **expectation**, of a discrete random variable of a probability distribution is the theoretical average of the variable.

- The expected value is, by definition, the mean of the probability distribution.

$$E(X) = \mu = \sum X \cdot P(X)$$

Ex.) 7 – Expectation

Dice Game A person pays $2 to play a certain game by rolling a single die once. If a 1 or a 2 comes up, the person wins nothing. If, however, the player rolls a 3, 4, 5, or 6, he or she wins the difference between the number rolled and $2. Find the expectation for this game. Is the game fair?

X	1	2	3	4	5	6
Win	$0	$0	$1	$2	$3	$4
$P(X)$	1/6	1/6	1/6	1/6	1/6	1/6

$$E(X) = \mu = \sum X \cdot P(X)$$

Ex.) 7 – Mean, Variance, Standard Deviation, and Expectation

We are interested in winnings, so we need to use 'Win' as our X:

X	P(X)	X*P(X)
0	1/6	0
0	1/6	0
1	1/6	1/6
2	1/6	2/6
3	1/6	3/6
4	1/6	4/6
Sum of X*P(X)		**10/6 = 1.667**

So our expectation, $E(x)=\mu$, is the Sum of X*P(X) which is:

$$\$1.667$$

However, we paid $2 to play the game. So we must subtract that from our calculation, and this will give us the true expectation:

$$E(X) = \$1.667 - \$2 = -\$0.33$$

Since this value is negative, it is not a fair game.

Expectation

- A box contains ten $1 bills, five $2 bills, three $5 bills, one $10 bill, and one $100 bill. A person is charged $20 to select one bill. Find the expected value for this game. Is this game fair?

5-3 The Binomial Distribution

- Many types of probability problems have only two possible outcomes or they can be reduced to two outcomes.
- Examples include: when a coin is tossed it can land on heads or tails, when a baby is born it is either a boy or girl, etc.

The Binomial Distribution

The **binomial experiment** is a probability experiment that satisfies these requirements:

1. Each trial can have only two possible outcomes—success or failure.
2. There must be a fixed number of trials.
3. The outcomes of each trial must be independent of each other.
4. The probability of success must remain the same for each trial.

The Binomial Distribution

Determine in which of the following situations a binomial distribution can be applied.

- Linda is interested in toilet paper pulling preferences. She takes a simple random sample of 5 people and asks each whether they always pull from the top or not. The probability that a person pulls from the top is 0.53, and X= the number of people who pull from the top.
- I roll a fair, 6-sided die until I get a two. X is the number of rolls it takes before I obtain a roll of two.
- You have a bag containing 4 red chips and 6 white chips and you draw 4 chips. Let random variable Y be the number of red chips drawn from the bag out of 4 draws without replacement.

Class Activity

- Which of the following are binomial distributions? Explain each answer.
  - Asking 100 students if they ate lunch today.
  - Asking the students in your class how they got to school today.
  - Drawing a club from a deck of cards.
  - Rolling a die to see the outcome.
  - Eating 3 different brands of hamburgers to find the favorite one.
  - Tossing a coin until you get a head.
  - Surveying 1000 students to see if they have a dog.

Notation for the Binomial Distribution

p	The symbol for the probability of success
q	The symbol for the probability of failure
n	The number of trials
X	The number of successes

Note that X = 0, 1, 2, 3,...,n

The Binomial Distribution

In a binomial experiment, the probability of exactly X successes in n trials is

$$P(X) = \frac{n!}{(n-X)!\,X!} \cdot p^{X} \cdot q^{n-X}$$

$$or$$

$$P(X) = \binom{n}{x} \cdot p^{X} \cdot q^{n-X}$$

The Binomial Probability Distribution

Notes

- 1. n! (*"n factorial"*) is an abbreviation for the product of the sequence of integers starting with n and ending with one. For example, 3! = 3 • 2 • 1 = 6 and 5! = 5 • 4 • 3 • 2 • 1 = 120. There is one special case, 0!, that is defined to be 1. For more information about **factorial notation**, see the *Student Solutions Manual*.
- 2. The values for n! and $\binom{n}{x}$ can be readily found using most scientific calculators.

The Binomial Probability Distribution

- 3. The binomial coefficient $\binom{n}{x}$ is equivalent to the number of combinations ${}_nC_x$, the symbol most likely on your calculator.
- 4. See the *Student Solutions Manual* for general information on the binomial coefficient.

Ex.) 9 – The Binomial Distribution

Compute the probability of X successes, using the binomial formula.

a. $n = 6, X = 3, p = 0.03$

$$P(X) = \frac{n!}{(n-X)!X!} \cdot p^X \cdot q^{n-X}$$

$$P(3) = \frac{6!}{3!3!} \cdot (0.03)^3 (1 - 0.03)^3$$

$$= \boxed{0.0005}$$

Ex.) 10 – The Binomial Distribution

Survey on Concern for Criminals In a survey, 3 of 4 students said the courts show "too much concern" for criminals. Find the probability that at most 3 out of 7 randomly selected students will agree with this statement.

$$P(X) = \frac{n!}{(n-X)!X!} \cdot p^X \cdot q^{n-X}$$

$$n = 7,\ p = \frac{3}{4},\ X = 0,1,2,3$$

Ex.) 10 – The Binomial Distribution

$$n = 7,\ p = \frac{3}{4},\ X = 0,1,2,3 \qquad P(X) = \frac{n!}{(n-X)!\,X!} \cdot p^{X} \cdot q^{n-X}$$

$$P(0) = \frac{7!}{7!0!} \cdot \left(\frac{3}{4}\right)^{0} \left(\frac{1}{4}\right)^{7} = 0.00006 \qquad P(1) = \frac{7!}{6!1!} \cdot \left(\frac{3}{4}\right)^{1} \left(\frac{1}{4}\right)^{6} = 0.00128$$

$$P(2) = \frac{7!}{5!2!} \cdot \left(\frac{3}{4}\right)^{2} \left(\frac{1}{4}\right)^{5} = 0.01154$$

$$P(3) = \frac{7!}{4!3!} \cdot \left(\frac{3}{4}\right)^{3} \left(\frac{1}{4}\right)^{4} = 0.05768$$

$$P(X \le 3) = P(0) + P(1) + P(2) + P(3)$$

$$= 0.00006 + 0.00128 + 0.01154 + 0.05768$$

$$\boxed{= 0.071}$$

The Binomial Distribution

1. Ryan is taking a twenty question true-false exam and plans to guess on each problem. Find the probability that he will get exactly 16 of the twenty questions correct.

Class Activity

- Stephen is taking a twenty question multiple choice test. Each question has 5 choices, A,B,C,D or E. Only one of the five is correct. If Stephen guesses on every problem, what is the probability that he will get exactly 8 correct?

The Binomial Distribution

- Suppose that 30% of the vehicles in a mall parking lot belong to employees. Nine vehicles are chosen at random. Find the probability that:
  - **Exactly 3** belong to mall employees.
  - **At most 3** belong to mall employees.
  - **At least 7** belong to mall employees.

Class Activity

- A survey indicates that 23% of US men select fishing as their favorite leisure activity.
  - Is this a binomial distribution?
  - If you randomly select 5 men, find the probability that exactly two of the men liked fishing.
  - Again, if you randomly select 5 men, find the probability that at least 2 of the men liked fishing.

The Binomial Distribution

The mean, variance, and standard deviation of a variable that has the *binomial distribution* can be found by using the following formulas.

$$\text{Mean: } \mu = np$$

$$\text{Variance: } \sigma^2 = npq$$

$$\text{Standard Deviation: } \sigma = \sqrt{npq}$$

Ex.) 11 – The Binomial Distribution

Defective Calculators If 3% of calculators are defective, find the mean, variance, and standard deviation of a lot of 300 calculators.

$$\mu = np = 300(0.03) = \boxed{9}$$

$$\sigma^2 = npq = 300(0.03)(0.97) = \boxed{8.73}$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{8.73} = \boxed{2.95}$$

The Binomial Distribution

- Linda is interested in toilet paper pulling preferences. She takes a simple random sample of 5 people and asks each whether they always pull from the top or not. The probability that a person pulls from the top is 0.53, and X= the number of people who pull from the top. Find the mean and standard deviation of X.

Class Activity

- According to United Mileage Plus Visa, 41% of passengers say they "put on the earphones" to avoid being bothered by their seatmates during flights. To show how important, or not important, the earphones are to people, consider the variable x to be the number of people in a sample of 12 who say they "put on the earphones" to avoid their seatmates. Assume the 41% is true for the whole population of airline travelers and that a random sample is selected. Find the mean and standard deviation of x.